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the λ's are all different, then $n - k\alpha$ will be equal to unity, and y_1 will be the only known integral. The number of known integrals will of course depend upon the number of λ's which are equal to λ_1. It is easy to find the order of the equation which can be no further reduced. We know that λ_1 is the greatest of the λ's, at least it is not less than any succeeding λ. By continuing the process of reduction long enough we must come to an equation of which

$$y_1 \text{ and } \frac{dy_1}{dx}$$

are integrals; other known integrals such as y_2, $\frac{dy_2}{dx}$, y_3, $\frac{dy_3}{dx}$, etc., may or may not exist according to the relative values of the λ's. The equation having y_1 and $\frac{dy_1}{dx}$ for integrals is readily found to be

$$\frac{d^{n-\lambda_1+2}y}{dx^{n-\lambda_1+2}} + q_1 \frac{d^{n-\lambda_1+1}y}{dx^{n-\lambda_1+1}} + \ldots + q_{n-\lambda_1+2}y = 0.$$

The coefficients q_1, $q_2 \ldots q_{n-\lambda_1+2}$ are of course derived by known processes from p_{11}, p_{12}, $\ldots p_{1n}$. As we know at least two integrals of this equation, viz. y_1 and $\frac{dy_1}{dx}$, we can reduce it one degree more, giving

$$\frac{d^{n-\lambda_1+1}y}{dx^{n-\lambda_1+1}} + r_1 \frac{d^{n-\lambda_1}y}{dx^{n-\lambda_1}} + \ldots + r_{n-\lambda_1+1}y = 0.$$

The degree of this equation can no longer be reduced, at least in the sense in which the word 'reduction' has been heretofore employed. We have then $n - k = n - \lambda_1 + 1$, or $k = \lambda_1 - 1$, *i. e.* $n - \lambda_1 + 1$ is the degree of the equation of the lowest order which can be obtained by the above process of reduction. Of course if we make the substitution

$$y = y_1 \int z dx$$

we find an equation in z of order $n - \lambda_1$, *none* of whose integrals are known.

Suppose now that the coefficients

$$p_{11}, p_{12}, \ldots p_{1n}$$

of the original equation are uniform simply periodic functions of the first kind having ω for period. Then one or more of the integrals will be uniform periodic functions of the second kind having for multipliers the different roots of the fundamental equation. If the roots of the fundamental equation are all simple, then we have as fundamental integrals of the given differential equation n uniform periodic functions of the second kind each possessing its own multiplier, which is different from all the others.

Suppose $\Delta = 0$ to be the fundamental equation, and $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_t$ its distinct roots, and $\nu_1, \nu_2, \ldots \nu_t$ the orders parting from which the minors of Δ cease to be all zero for $\varepsilon = \varepsilon_i\,(i = 1, 2, \ldots t)$; then it is known* that the differential equation admits $\nu_1 + \nu_2 + \ldots + \nu_t$ distinct integrals, which are uniform periodic functions of the second kind. Again, it is known that, in order that the differential equation may admit of n integrals which are periodic functions of the second kind, it is necessary and sufficient that each root of the fundamental equation $\Delta = 0$ shall annul all the minors of Δ up to the order equal to the degree of multiplicity of the root. Finally, those integrals of the equation which are not periodic functions of the second kind are of the form of polynomials in x, the coefficients of the different powers of x being periodic functions of the second kind, all having the same multiplier. In the case in hand we have the differential equation

$$\frac{d^n y}{dx^n} + p_{11}\frac{d^{n-1}y}{dx^{n-1}} + p_{12}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_{1n}y = 0,$$

the coefficients of which are uniform periodic functions of the first kind, having ω for period. The integrals of the equation are

$$\begin{array}{l}
y_1, \dfrac{dy_1}{dx}, \dfrac{d^2y_1}{dx^2}, \ldots \dfrac{d^{\lambda_1-1}y_1}{dx^{\lambda_1-1}}, \\
y_2, \dfrac{dy_2}{dx}, \dfrac{d^2y_2}{dx^2}, \ldots \dfrac{d^{\lambda_2-1}y_2}{dx^{\lambda_2-1}}, \\
\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\
y_\alpha, \dfrac{dy_\alpha}{dx}, \dfrac{d^2y_\alpha}{dx^2}, \ldots \dfrac{d^{\lambda_\alpha-1}y_\alpha}{dx^{\lambda_\alpha-1}}.
\end{array}$$

Since the derivatives of $y_1, y_2, \ldots y_\alpha$ must be functions of the same kind as the y's themselves, it is clear that the fundamental equation cannot have more than α distinct roots. If the equation have α distinct roots $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_\alpha$, these will of course be the multipliers of $y_1, y_2, \ldots y_\alpha$, and consequently of their derivatives; *i. e.* in this case the equation has n distinct integrals which are periodic functions of the second kind. The degrees of multiplicity of the roots are respectively $\lambda_1, \lambda_2, \ldots \lambda_\alpha$, and obvious, ε_1 makes each minor of Δ up to the order λ_1 vanish, ε_2 makes each minor vanish up to the order λ_2, and so on. The case when the fundamental equation has fewer than α roots is included in the above general theorem; as an illustration, however, suppose that the equation has roots

$$\varepsilon_1, \varepsilon_2, \ldots \varepsilon_\beta, \qquad \beta < \alpha$$

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* Floquet: Annales de l'École Normale, February, 1883, page 48.

belonging to the functions

$$y_1, y_2, \dots y_\beta .$$

Then it is obvious that we have

$$\lambda_1 + \lambda_2 + \dots + \lambda_\beta$$

integrals, which are periodic functions of the second kind.

Similar remarks apply of course in the case of coefficients which are doubly periodic functions of the first kind, except that in this case we have two fundamental equations corresponding to the two sets of multipliers.

Resume now the original system of equations of the n^{th} order, satisfied by y_i and its $\lambda_i - 1$ derivatives, viz.

$$\begin{aligned}
&\frac{d^n y_i}{dx^n} + p_{11} \frac{d^{n-1} y_i}{dx^{n-1}} + \dots + p_{1n} y = 0, \\
&\frac{d^{n+1} y_i}{dx^{n+1}} + p_{11} \frac{d^n y_i}{dx^n} + \dots + p_{1n} \frac{dy}{dx} = 0, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\frac{d^{n+\lambda_i-1} y_i}{dx^{n+\lambda_i-1}} + p_{11} \frac{d^{n+\lambda_i-2} y_i}{dx^{n+\lambda_i-2}} + \dots + p_{1n} \frac{d^{\lambda_i-1} y}{dx^{\lambda_i-1}} = 0.
\end{aligned}$$

From the first and third of these we derive after easy reductions the equation

$$p''_{11} y_i^{(n-1)} + p''_{12} y_i^{(n-2)} + \dots + p''_{1n} y_i = 0,$$

or, writing

$$\frac{p''_{12}}{p''_{11}} = p^{(2)}_{22}, \quad \frac{p''_{13}}{p''_{11}} = p^{(2)}_{23} \dots,$$

$$y_i^{(n-1)} + p^{(2)}_{22} y_i^{(n-2)} + \dots + p^{(2)}_{2n} y = 0;$$

similarly

$(P_i)_2$

$$\begin{aligned}
&y_i^{(n)} + p^{(2)}_{22} y_i^{(n+1)} + \dots + p^{(2)}_{2n} y' = 0, \\
&y_i^{(n+1)} + p^{(2)}_{22} y_i^{(n)} + \dots + p^{(2)}_{2n} y_i'' = 0, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&y_i^{(n+\lambda_i-3)} + p^{(2)}_{22} y_i^{(n+\lambda_i-4)} + \dots + p^{(2)}_{2n} y^{(\lambda_i-3)} = 0.
\end{aligned}$$

The integrals of the equation

$P^{(2)}$

$$\frac{d^{n-1} y}{dx^{n-1}} + p^{(2)}_{22} \frac{d^{n-2} y}{dx^{n-2}} + \dots + p^{(2)}_{2n} y = 0$$

are easily seen to be

$$\begin{aligned}
&y_1, \frac{dy_1}{dx}, \frac{d^2 y_1}{dx^2} \dots \frac{d^{\lambda_1-3} y_1}{dx^{\lambda_1-3}}, \\
&y_2, \frac{dy_2}{dx}, \frac{d^2 y_2}{dx^2} \dots \frac{d^{\lambda_2-3} y_2}{dx^{\lambda_2-3}}, \\
&\dots\dots\dots\dots\dots\dots \\
&y_\alpha, \frac{dy_\alpha}{dx}, \frac{d^2 y_\alpha}{dx^2} \dots \frac{d^{\lambda_\alpha-3} y_\alpha}{dx^{\lambda_\alpha-3}}.
\end{aligned}$$

We have thus $n - 2\alpha$ integrals of the equation of the order $n - 1$, and these are the same as the known $n - 2\alpha$ integrals of the equation

$$P^{(1)\prime} \qquad p_{22}^{(1)\prime} y^{(n-2)} + p_{23}^{(1)\prime} y^{(n-3)} + \ldots + p_{2n}^{(1)\prime} y = 0,$$

or say

$$y^{(n-2)} + p_{33}^{(1)} y^{(n-3)} + \ldots + p_{3n}^{(1)} y = 0.$$

There remain now $2\alpha - 1$ unknown integrals of $P^{(2)}$ and $2(\alpha - 1)$ unknown integrals of $P^{(1)\prime}$.

From $(P_i)_2$ we can form again the equation

$$p_{22}^{(2)\prime} y^{(n-2)} + p_{23}^{(2)\prime} y^{(n-3)} + \ldots + p_{2n}^{(2)\prime} y = 0,$$

or say

$$y^{(n-2)} + p_{33}^{(2)} y^{(n-3)} + \ldots + p_{3n}^{(2)} y = 0,$$

the integrals which are easily seen to be

$$\begin{array}{l} y_1, \dfrac{dy_1}{dx}, \dfrac{d^2y_1}{dx^2} \ldots \dfrac{d^{\lambda_1-4}y_1}{dx^{\lambda_1-4}}, \\ y_2, \dfrac{dy_2}{dx}, \dfrac{d^2y_2}{dx^2} \ldots \dfrac{d^{\lambda_2-4}y_2}{dx^{\lambda_2-4}}, \\ \ldots\ldots\ldots\ldots\ldots \\ y_\alpha, \dfrac{dy_\alpha}{dx}, \dfrac{d^2y_\alpha}{dx^2} \ldots \dfrac{d^{\lambda_\alpha-4}y_\alpha}{dx^{\lambda_\alpha-4}}. \end{array}$$

These $n - 3\alpha$ integrals are the known integrals of

$$y^{(n-3)} + p_{44}^{(1)} y^{(n-4)} + \ldots + p_{4n}^{(1)} y = 0$$

mentioned above. There remain now $3\alpha - 2$ unknown integrals of the new equation of order $n - 2$ and $3(\alpha - 1)$ integrals of the equation of order $n - 3$.

This process can of course be continued just as in the earlier case, where the coefficients were $p_{ij}^{(1)}$; it is hardly worth while, however, saying anything more about it.

We may go back, however, to the original system once more, and by combination of the first and fourth, the second and fifth, etc., and making some easy reductions, obtain

$$\begin{array}{l} y_i^{(n-1)} + p_{22}^{(3)} y^{(n-2)} + \ldots + p_{2n}^{(3)} y = 0, \\ y_i^{(n-1)} + p_{22}^{(3)} y^{(n-2)} + \ldots + p_{2n}^{(3)} y' = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_i^{(n+\lambda_i-4)} + p_{22}^{(3)} y^{(n+\lambda_i-5)} + \ldots + p_{2n}^{(3)} y^{(\lambda_i-4)} = 0, \end{array}$$

where

$$p_{22}^{(3)} = p_{12}''' \div p_{11}''', \quad p_{23}^{(3)} = p_{13}''' \div p_{11}''', \text{ etc.}$$

The integrals of

$$\frac{d^{n-1}y}{dx^{n-1}} + p_{22}^{(3)} \frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_{2n}^{(3)} y = 0$$

are obviously

$$\begin{array}{l} y_1,\ \frac{dy_1}{dx},\ \frac{d^2y_1}{dx^2},\ \ldots\ \frac{d^{\lambda_1-4}y_1}{dx^{\lambda_1-4}}, \\ y_2,\ \frac{dy_2}{dx},\ \frac{d^2y_2}{dx^2},\ \ldots\ \frac{d^{\lambda_2-4}y_2}{dx^{\lambda_2-4}}, \\ \cdots\cdots\cdots\cdots\cdots \\ y_\alpha,\ \frac{dy_\alpha}{dx},\ \frac{d^2y_\alpha}{dx^2},\ \ldots\ \frac{d^{\lambda_\alpha-4}y_\alpha}{dx^{\lambda_\alpha-4}}. \end{array}$$

We know thus $n - 3\alpha$ integrals of

$$y^{(n-1)} + p_{22}^{(3)} y^{(n-2)} + \ldots + p_{2n}^{(3)} y = 0,$$

and these are the known integrals of the equations

$$y^{(n-2)} + p_{33}^{(2)} y^{(n-3)} + \ldots + p_{3n}^{(2)} y = 0,$$

$$y^{(n-3)} + p_{44}^{(1)} y^{(n-4)} + \ldots + p_{4n}^{(1)} y = 0.$$

There remain then $3\alpha - 3$ integrals of the equation of order $n - 3$, $3\alpha - 2$ unknown integrals of the equation of order $n - 2$, and $3\alpha - 1$ unknown integrals of the equation of order $n - 1$.

Taking the general case, combine the first of the original set with the $i + 1$st, the second with the $i + 2$nd, etc.; we have as the result

$$y^{(n-1)} + p_{22}^{(i)} y^{(n-2)} + \ldots + p_{2n}^{(i)} y = 0,$$

where

$$p_{22}^{(i)} = \frac{d^i p_{12}}{dx^i} \div \frac{d^i p_{11}}{dx^i},\quad p_{23}^{(i)} = \frac{d^i p_{13}}{dx^i} \div \frac{d^i p_{11}}{dx^i},\ \text{etc.}$$

The known integrals of this are

$$\begin{array}{l} y_1,\ \frac{dy_1}{dx},\ \frac{d^2y_1}{dx^2},\ \ldots\ \frac{d^{\lambda_1-i-1}y_1}{dx^{\lambda_1-i-1}}, \\ y_2,\ \frac{dy_2}{dx},\ \frac{d^2y_2}{dx^2},\ \ldots\ \frac{d^{\lambda_2-i-1}y_2}{dx^{\lambda_2-i-1}}, \\ \cdots\cdots\cdots\cdots\cdots \\ y_\alpha,\ \frac{dy_\alpha}{dx},\ \frac{d^2y_\alpha}{dx^2},\ \ldots\ \frac{d^{\lambda_\alpha-i-1}y_\alpha}{dx^{\lambda_\alpha-i-1}}. \end{array}$$

These are the known integrals now of the set of equations, including the $p^{(i)}$ equation,

$$\begin{array}{llll} y^{(n-1)} + p_{22}^{(i)} y^{(n-2)} & + \ldots + p_{2n}^{(i)} y & = 0, \\ y^{(n-2)} + p_{33}^{(i-1)} y^{(n-3)} & + \ldots + p_{3n}^{(i-1)} y & = 0, \\ y^{(n-3)} + p_{44}^{(i-2)} y^{(n-4)} & + \ldots + p_{4n}^{(i-2)} y & = 0, \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots & \cdots \\ y^{(n-i)} + p_{i+1,\,i+1}^{(1)} y^{(n-i-1)} & + \ldots + p_{i+1,\,n}^{(1)} y & = 0. \end{array}$$

The numbers of unknown integrals of these equations are respectively

$$i\alpha - 1,\ i\alpha - 2,\ i\alpha - 3,\ \ldots\ i\alpha - i.$$

In résumé: we have the two equations

$$(P) = \frac{d^n y}{dx^n} + p_{11}\frac{d^{n-1}y}{dx^{n-1}} + p_{12}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_{1n}y = 0,$$

$$\frac{d^{n-1}y}{dx^{n-1}} + p_{22}^{(1)}\frac{d^{n-2}y}{dx^{n-2}} + p_{23}^{(1)}\frac{d^{n-3}y}{dx^{n-3}} + \ldots + p_{2n}y = 0,$$

having the common $n - \alpha$ integrals

$$\begin{gathered} y_1, \frac{dy_1}{dx} \ldots \frac{d^{\lambda_1 - 2}y_1}{dx^{\lambda_1 - 2}}, \\ \ldots\ldots\ldots\ldots \\ y_\alpha, \frac{dy_\alpha}{dx} \ldots \frac{d^{\lambda_\alpha - 2}y_\alpha}{dx^{\lambda_\alpha - 2}}; \end{gathered}$$

the three equations

$$\begin{gathered} (P) = 0, \\ \frac{d^{n-1}y}{dx^{n-1}} + p_{22}^{(2)}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_{2n}^{(2)}y = 0, \\ \frac{d^{n-2}y}{dx^{n-2}} + p_{33}^{(1)}\frac{d^{n-3}y}{dx^{n-3}} + \ldots + p_{3n}^{(1)}y = 0, \end{gathered}$$

having the $n - 2\alpha$ common integrals

$$\begin{gathered} y_1, \frac{dy_1}{dx} \ldots \frac{d^{\lambda_1 - 3}y_1}{dx^{\lambda_1 - 3}}, \\ \ldots\ldots\ldots\ldots \\ y_\alpha, \frac{dy_\alpha}{dx} \ldots \frac{d^{\lambda_\alpha - 3}y_\alpha}{dx^{\lambda_\alpha - 3}}, \end{gathered}$$

and the $i + 1$ equations

$$\begin{aligned} &(P) = 0 \\ &\frac{d^{n-1}y}{dx^{n-1}} + p_{22}^{(i)}\frac{d^{n-2}y}{dx^{n-2}} \qquad + \ldots + p_{2n}^{(i)}y \qquad = 0, \\ &\frac{d^{n-2}y}{dx^{n-2}} + p_{33}^{(i-1)}\frac{d^{n-3}y}{dx^{n-3}} \qquad + \ldots + p_{3n}^{(i-1)}y = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\frac{d^{n-i}y}{dx^{n-i}} + p_{i+1,\,i+1}^{(1)}\frac{d^{n-i-1}y}{dx^{n-i-1}} + \ldots + p_{i+1,\,n}^{(1)}y = 0, \end{aligned}$$

having the $n - i\alpha$ common integrals

$$\begin{gathered} y_1, \frac{dy_1}{dx} \ldots \frac{d^{\lambda_1 - i - 1}y_1}{dx^{\lambda_1 - i - 1}}, \\ \ldots\ldots\ldots\ldots \\ y_\alpha, \frac{dy_\alpha}{dx} \ldots \frac{d^{\lambda_\alpha - i - 1}y_\alpha}{dx^{\lambda_\alpha - i - 1}}. \end{gathered}$$

From each of these systems we may deduce an equation satisfied only by the common integrals, *i. e.* we would have, including the given equation, a new system of equations of the respective degrees n, $n - \alpha$, $n - 2\alpha, \ldots n - i\alpha \ldots$ satisfied only by the corresponding group of common integrals. The properties of each of the new equations will of course be exactly the same as those of the original equation.

In all that precedes it has been tacitly assumed, unless statement is made to the contrary, that all the functions $y_1, y_2, \ldots y_\alpha$, and consequently their derivatives, are uniform functions; further, if a_i denote a singular point of the function, the function will remain finite at the point a_i on being multiplied by a determinate finite power of $(x - a_i)$. The coefficients $p_{11}, p_{12}, \ldots p_{1n}$ are therefore of the form

$$\frac{P_{11}}{x - a_i}, \frac{P_{12}}{(x - a_i)^2}, \ldots \frac{P_{1n}}{(x - a_i)^n}$$

where

$$P_{1k} = \frac{F_{k(\rho-1)}(x)(x - a_i)^k}{\psi^k}$$

ρ denoting the total number of singular points; $F_m(x)$ an integral function of x at most of the degree m; and $\psi = (x - a_1)(x - a_2) \ldots (x - a_\rho)$.*

It is of course perfectly obvious that the coefficients of the various derived equations are of the same form as those of the given equation, formed as they are merely by differentiation, multiplication and division. This being the case it follows that the unknown integrals of these derived equations are uniform functions of the same character as the known integrals

$$\begin{array}{l} y_1, \dfrac{dy_1}{dx}, \dfrac{d^2y_1}{dx^2}, \ldots \dfrac{d^{k_1}y_1}{dx^k}, \\ y_2, \dfrac{dy_2}{dx}, \dfrac{d^2y_2}{dx^2}, \ldots \dfrac{d^{k_2}y_2}{dx^{k_2}}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ y_\alpha, \dfrac{dy_\alpha}{dx}, \dfrac{d^2y_\alpha}{dx^2}, \ldots \dfrac{d^{k_\alpha}y_\alpha}{dx^{k_\alpha}}. \end{array}$$

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* Crelle, Vol. LXVI. Fuchs: Zur Theorie der linearen Differentialgleichungen mit veränderlichen Coefficienten.